CHAPTER 4

IMAGE ENHANCEMENT IN THE FREQUENCY DOMAIN

CHAPTER 4: IMAGE ENHANCEMENT IN THE FREQUENCY DOMAIN

- French mathematician J. B. J. Fourier was born in 1768.
- He died of heart disease at the age of 59.
- **Fourier Series**: Any periodic function can be expressed as the sum of complex exponentials (sines and/or cosines) of different frequencies, each multiplied by a different coefficient.
  - *f(x)*: single variable, continuous function with period T_0

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{j\Omega_0 kx}, \Omega_0 = 2\pi / T_0$$

- **Fourier Transform**: Non-periodic functions can also be represented in terms of complex exponentials.
  - *f*(*x*): single variable, continuous function.

$$\underbrace{F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx \qquad f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du}_{\text{Fourier transform pair}}$$

FOURIER SERIES

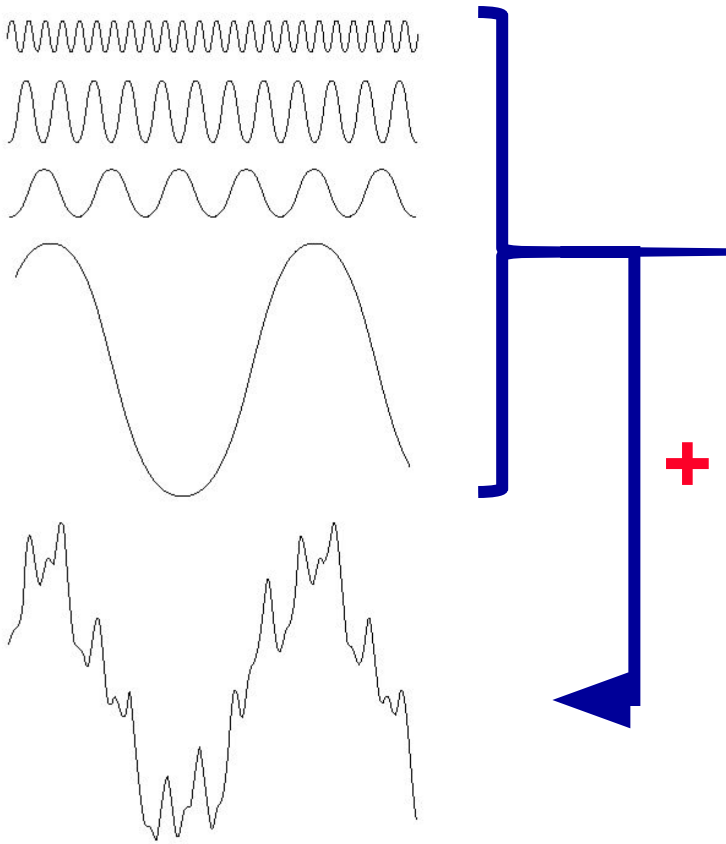


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

FOURIER TRANSFORM IN 2 VARIABLES

Fourier Transform can easily be extended to 2 variables:

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}dudv$$

ONE-DIMENSIONAL DFT AND ITS INVERSE

$f(x), x = 0,1,2,\ldots, M-1$: **discrete function of one variable**

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M}, u = 0,1,\ldots, M-1$$

Approximately M^2 summations and multiplications to compute

$$f(x) = \sum_{u=0}^{M-1} F(u)e^{j2\pi ux/M}, x = 0,1,\ldots, M-1$$

The DFT and its inverse always exist.

FREQUENCY DOMAIN & FREQUENCY COMPONENTS

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\Rightarrow F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)[\cos 2\pi ux/M - j\sin 2\pi ux/M], u = 0,1,2,\ldots, M-1.$$

The domain over which the values of *F*(*u*) range is called the *frequency domain*.

Each of the *M* terms of *F*(*u*) is called a frequency component of the transform.

***u* & *F*(*u*): frequency domain & frequency components.**
***x* and *f*(*x*): time domain and time components.**

A useful analogy: Compare Fourier transform to a glass prism.

- **Glass prism: a physical device that separates light into its various color components.**
- **Fourier transform: a mathematical prism that separates a function into various components based on frequency content.**

FOURIER TRANSFORM IN POLAR COORDINATES

$$F(u) = |F(u)|\, e^{-j\phi(u)},$$

where $|F(u)| = [R^2(u) + I^2(u)]^{1/2}$ **and** $\phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$

magnitude

phase angle

$$P(u) = |F(u)|^2 = R^2(u) + I^2(u)$$

power spectrum

A ONE-DIMENSIONAL EXAMPLE

M = 1024
A = 1
K = 8

It is common practice to multiply $f(x)$ by $(-1)^x$ before taking the transform.

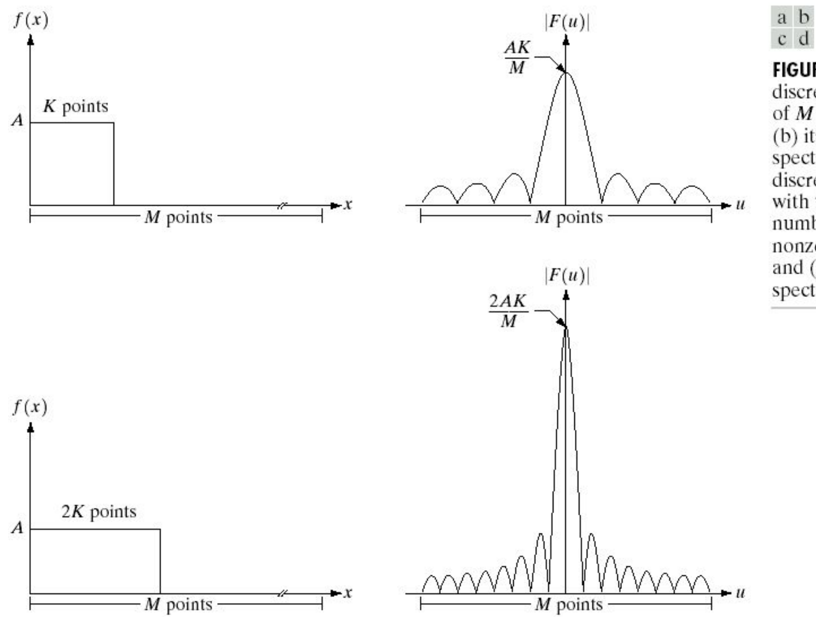


FIGURE 4.2 (a) A discrete function of M points, and (b) its Fourier spectrum. (c) A discrete function with twice the number of nonzero points, and (d) its Fourier spectrum.

Important things to note:

1. The height of the spectrum doubled as the area under the curve in the x-domain doubled.
2. The # of zeros in the spectrum in the same interval doubled as the length of the function doubled.

M = 1024
A = 1
K = 16

SAMPLES IN SPATIAL AND FREQUENCY DOMAINS

$f(x), x = 0,1,\ldots, M-1:$ *M* samples

$f(x_0):$ *first* point in the sequence

$$f(x_0 + \Delta x), f(x_0 + k\Delta x), \ldots, f(x_0 + [M-1]\Delta x)$$

$$f(x) = f(x_0 + x\Delta x)$$

$F(u), u = 0,1,\ldots, M-1:$ *M* samples

$0:$ *first* point in the sequence

$$F(0 + \Delta u), F(0 + k\Delta u), \ldots, F(0 + [M-1]\Delta u)$$

$$F(u) = F(u\Delta u)$$

$$\Delta u = \frac{1}{M\Delta x}$$

TWO DIMENSIONAL DFT AND ITS INVERSE

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}, \underbrace{u = 0,1,2,\ldots,M-1, v = 0,1,2,\ldots,N-1.}_{\text{frequency variables}}$$

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}, \underbrace{x = 0,1,2\ldots,M-1, y = 0,1,2,\ldots,N-1.}_{\text{spatial variables}}$$

Polar coordinates: $|F(u,v)| = [R^2(u,v) + I^2(u,v)]^{1/2}$ $\quad \phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$

Power spectrum: $P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v)$

SHIFTING OF ORIGIN

$$\Im[f(x,y)(-1)^{x+y}] = F(u - M/2, v - N/2)$$

Fourier Transform

This shifts the origin of $F(u,v)$ to $(M/2,N/2)$.

$$F(0,0) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$$

dc component

- **If $f(x,y)$ is an image, the value of DFT at the origin is equal to the average gray level of the image.**

$f(x,y)$ is real: $F(u,v) = F^*(-u,-v)$

- $|F(u,v)| = |F(-u,-v)|$

Spectrum is symmetric

$$\Delta u = \frac{1}{M\Delta x} \quad \text{and} \quad \Delta v = \frac{1}{M\Delta y}$$

CENTERING THE SPECTRUM

a b

FIGURE 4.3 (a) Image of a 20 × 40 white rectangle on a black background of size 512 × 512 pixels. (b) Centered Fourier spectrum shown after application of the log transformation given in Eq. (3.2-2). Compare with Fig. 4.2.

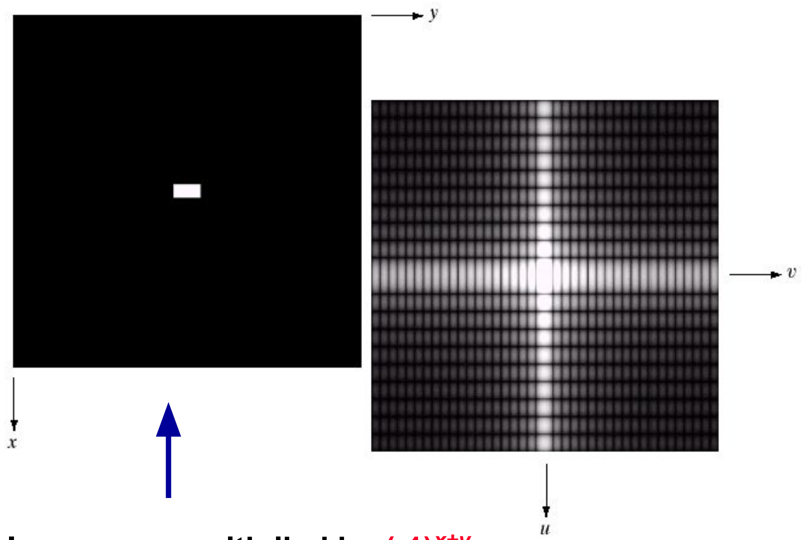


Image was multiplied by $(-1)^{x+y}$ prior to computing the transform.

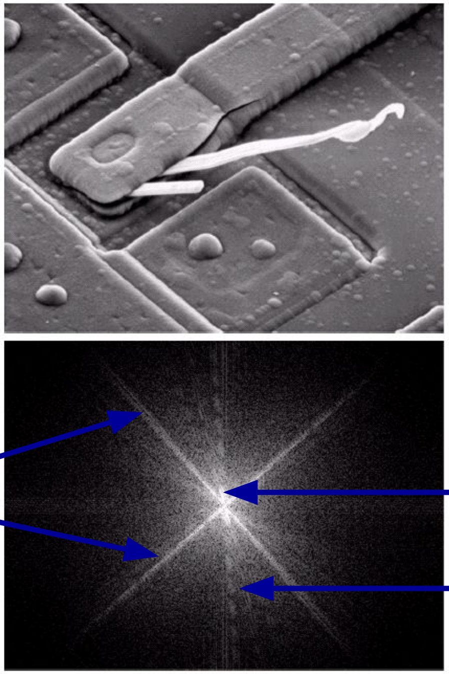
RELATIONSHIP BETWEEN THE DFT COMPONENTS AND SPATIAL CHARACTERISTICS OF AN IMAGE

Frequency is directly related to the rate of change.

DFT components can be associated with patterns of intensity variations in an image.

2 principal features:

1. **Strong edges that run approx. at ±45^0.**
2. **2 white oxide protrusions.**



a
b

FIGURE 4.4 (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

Correspond to the strong edges.

Corresponds to the short protrusion.

Corresponds to the long protrusion.

BASIC STEPS IN FREQUENCY DOMAIN FILTERING

$(-1)^{x+y}$
Cropping to even dimensions
Gray-level scaling
Conversion to floating point on input
Conversion to 8-bit format on output
Etc.

Each component of *H* multiplies both real and imaginary parts of the corresponding component of *F*. Such filters are called zero-phase-shift filters.

Imaginary components set to zero.

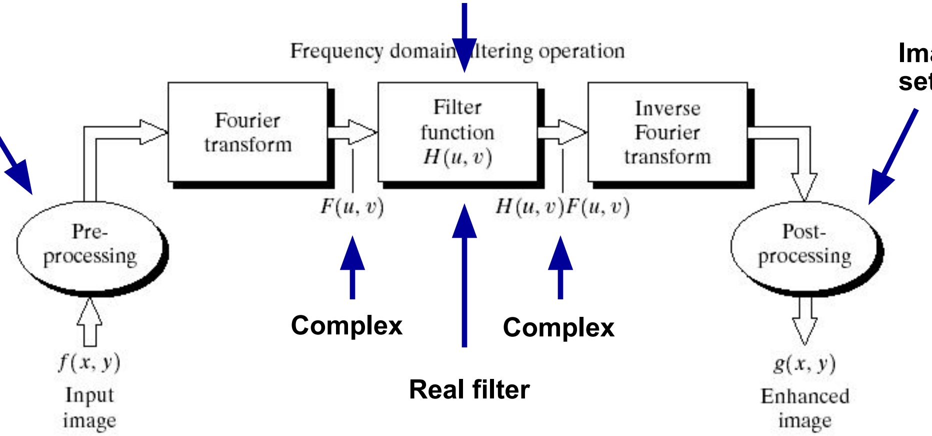


FIGURE 4.5 Basic steps for filtering in the frequency domain.

1. Multiply the input image by $(-1)^{x+y}$
2. Compute $F(u,v)$
3. Multiply $F(u,v)$ by a filter function $H(u,v)$
4. Compute the inverse DFT
5. Obtain the real part
6. Multiply the real part by $(-1)^{x+y}$

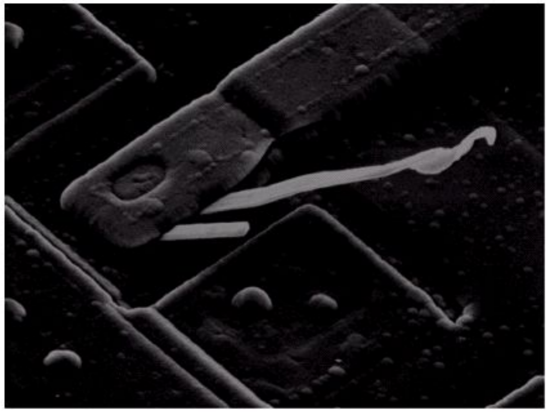
AN INTRODUCTORY EXAMPLE OF FILTERING

Assume we want the average value of an image to be zero. Also assume the transform has been centered.

- $F(0,0) = 0$
- $H(u,v) = \begin{cases} 0 & \text{if } (u,v) = (M/2, N/2) \\ 1 & \text{otherwise} \end{cases}$

notch filter: constant function with a notch at the origin

FIGURE 4.6 Result of filtering the image in Fig. 4.4(a) with a notch filter that set to 0 the $F(0, 0)$ term in the Fourier transform.



In reality, the average of the displayed image cannot be zero.

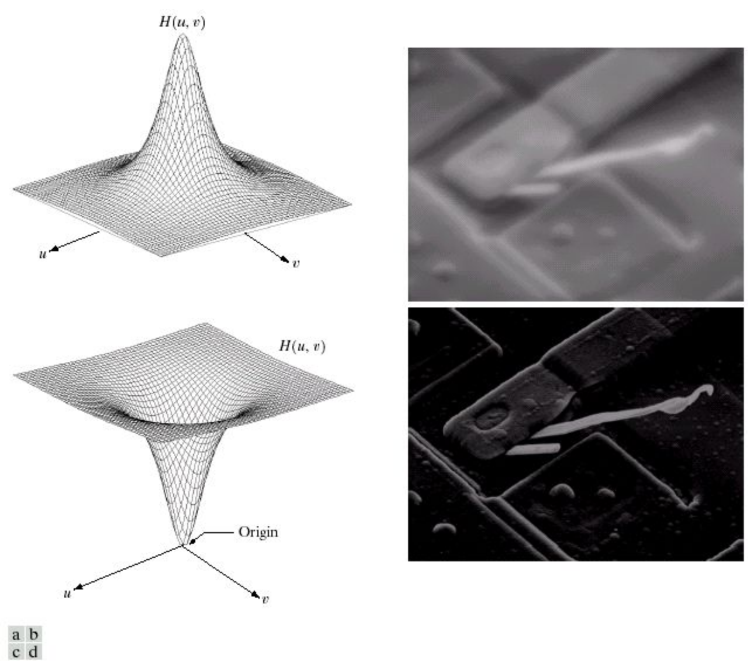
The display of this image is made possible by scaling (making the most negative value 0, and scaling all other values up from that).

LOW-PASS AND HIGH-PASS FILTERS

Low-pass filter: **A filter that attenuates high frequencies while passing low frequencies.**

High-pass filter: **A filter that attenuates low frequencies while passing high frequencies.**

Circularly symmetric



blurring

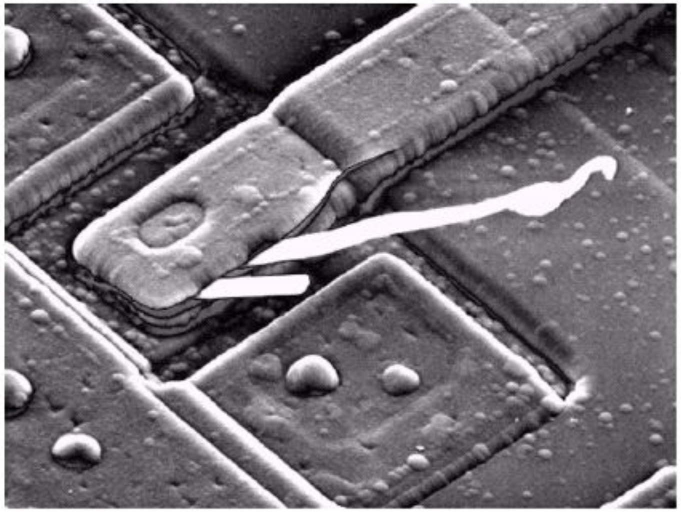
sharpening

FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).

ADDING A CONSTANT TO A HIGH-PASS FILTER

A <u>constant</u> is added to the HP filter so that it does not completely eliminate *F*(0,0).

FIGURE 4.8
Result of highpass filtering the image in Fig. 4.4(a) with the filter in Fig. 4.7(c), modified by adding a constant of one-half the filter height to the filter function. Compare with Fig. 4.4(a).



Notice the improvement!

CONVOLUTION

$$f(x,y)*h(x,y)=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f(m,n)h(x-m,y-n)$$

Discrete convolution of 2 functions

For each displacement (*x*,*y*), computes the sum of products over all values of *m* and *n*.

$$\left.\begin{aligned} f(x,y)*h(x,y) &\Leftrightarrow F(u,v)H(u,v) \\ f(x,y)h(x,y) &\Leftrightarrow F(u,v)*H(u,v) \end{aligned}\right\}\text{Convolution theorem}$$

IMPULSE FUNCTION OF STRENGTH A

An impulse function of strength *A* located at coordinates (x_0,y_0)

- $$A\delta(x-x_0, y-y_0)$$

Definition:

$$\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} s(x,y)A\delta(x-x_0, y-y_0) = As(x_0, y_0)$$

Unit impulse located at the origin:

$$\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} s(x,y)A\delta(x,y) = s(0,0)$$

DFT of the unit impulse located at the origin:

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} \delta(x,y)e^{-j2\pi(ux/M+vy/N)} = \frac{1}{MN}$$

real constant

FOURIER TRANSFORM PAIRS

$$f(x,y) * h(x,y) \Leftrightarrow F(u,v)H(u,v)$$

$$\delta(x,y) * h(x,y) \Leftrightarrow \Im[\delta(x,y)]H(u,v)$$

$$\frac{1}{MN}h(x,y) \Leftrightarrow \frac{1}{MN}H(u,v)$$

$$h(x,y) \Leftrightarrow H(u,v)$$

Hence, filters in the spatial and frequency domains constitute a Fourier transform pair.

If both filters are of the same size, it is more efficient to do the filtering in the frequency domain.

However, we use much smaller filters in the spatial domain.

GAUSSIAN FILTERS

$$H(u) = Ae^{-u^2/2\sigma^2}$$

$$h(x) = \sqrt{2\pi}\sigma A e^{-2\pi^2\sigma^2 x^2}$$

Fourier transform pair

Both are real.

Gaussian curves are intuitive and easy to manipulate.

They behave reciprocally w.r.t one another.

***H*(*u*) has a broad profile • *h*(*x*) has a narrow profile.**

***H*(*u*) has a narrow profile • *h*(*x*) has a broad profile.**

LOWPASS AND HIGHPASS GAUSSIAN FILTERS

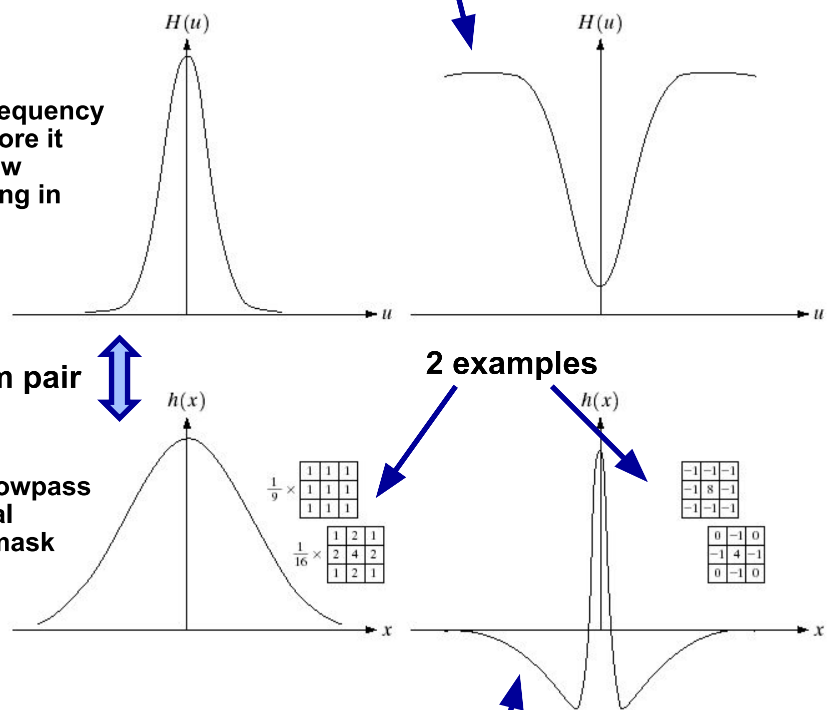
$$H(u) = Ae^{-u^2/2\sigma_1^2} - Be^{-u^2/2\sigma_2^2}$$

The narrower the frequency domain filter, the more it will attenuate the low frequencies, resulting in increased blurring.

Fourier transform pair

We can implement lowpass filtering in the spatial domain by using a mask with all positive coefficients.

2 examples



a b
c d

FIGURE 4.9 (a) Gaussian frequency domain lowpass filter. (b) Gaussian frequency domain highpass filter. (c) Corresponding lowpass spatial filter. (d) Corresponding highpass spatial filter. The masks shown are used in Chapter 3 for lowpass and highpass filtering.

The spatial filter has both negative and positive filters.

$$h(x) = \sqrt{2\pi}\sigma_1 Ae^{-2\pi^2\sigma_1^2x^2} - \sqrt{2\pi}\sigma_2 Be^{-2\pi^2\sigma_2^2x^2}$$

SMOOTHING AND SHARPENING FILTERS

- **Smoothing filters**
  - **Ideal lowpass filters (very sharp)**
  - **Butterworth lowpass filters**
  - **Gaussian lowpass filters (very smooth)**

Butterworth filter parameter: filter order

- **High values: filter has the form of the ideal filter.**
- **Low values: filter has the form of the Gaussian filter.**

- **Sharpening filters**
  - **Ideal highpass filters**
  - **Butterworth highpass filters**
  - **Gaussian highpass filters**

2-D IDEAL LOWPASS FILTERS

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$$

Nonnegative quantity

Distance from (u,v) to the center of the frequency rectangle

Image **size**: MxN

Center of the frequency rectangle: $(u,v) = (M/2, N/2)$

Distance to the center: $D(u,v) = [(u - M/2)^2 + (v - N/2)^2]^{1/2}$

The lowpass filters in this chapter are **radially symmetric**.

2-D IDEAL LOWPASS FILTERS

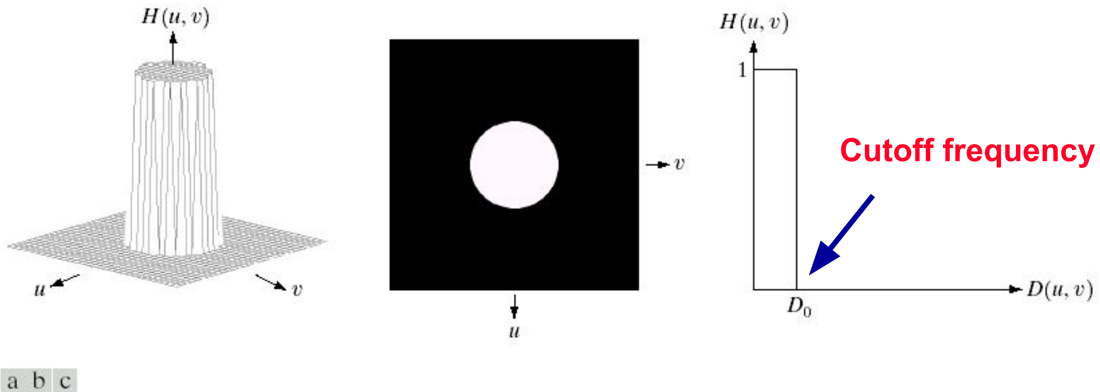


FIGURE 4.10 (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

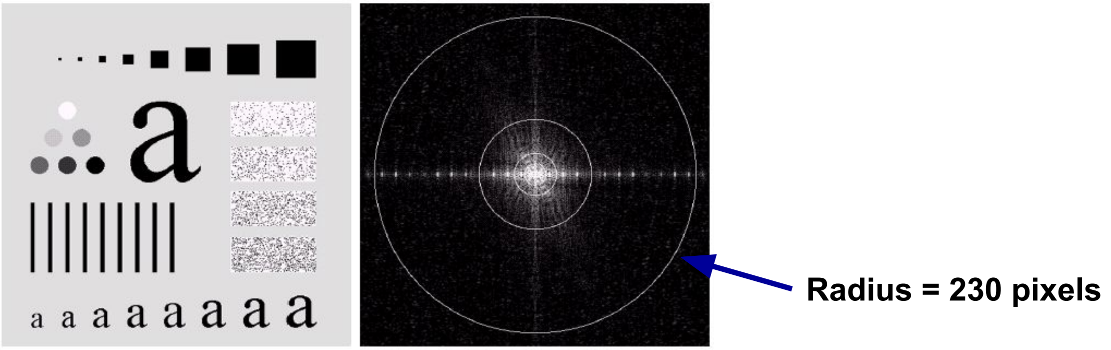
2-D IDEAL LOWPASS FILTERS AS A FUNCTION OF CUTOFF FREQUENCIES

Total image power:

$$P_T = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} P(u,v)$$

A circle of radius r encloses α percent of the power:

$$\alpha = 100\left[\sum_u \sum_v P(u,v) / P_T\right]$$



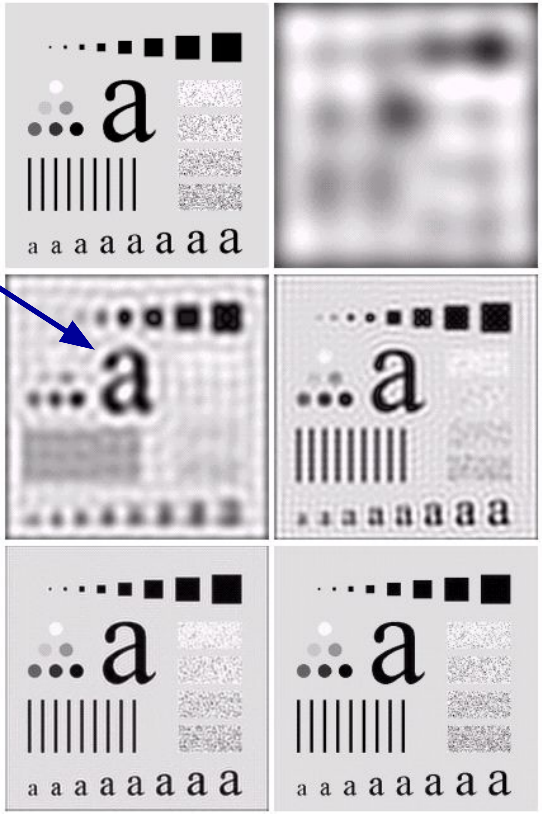
a b

FIGURE 4.11 (a) An image of size 500 × 500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.

ILP FILTERING WITH RADII 5,15,30,80,230

Ringing is characteristics of ideal filters

Ideal lowpass filtering is not very practical but they can be implemented on a computer to study their behavior.



Increase in filter radius

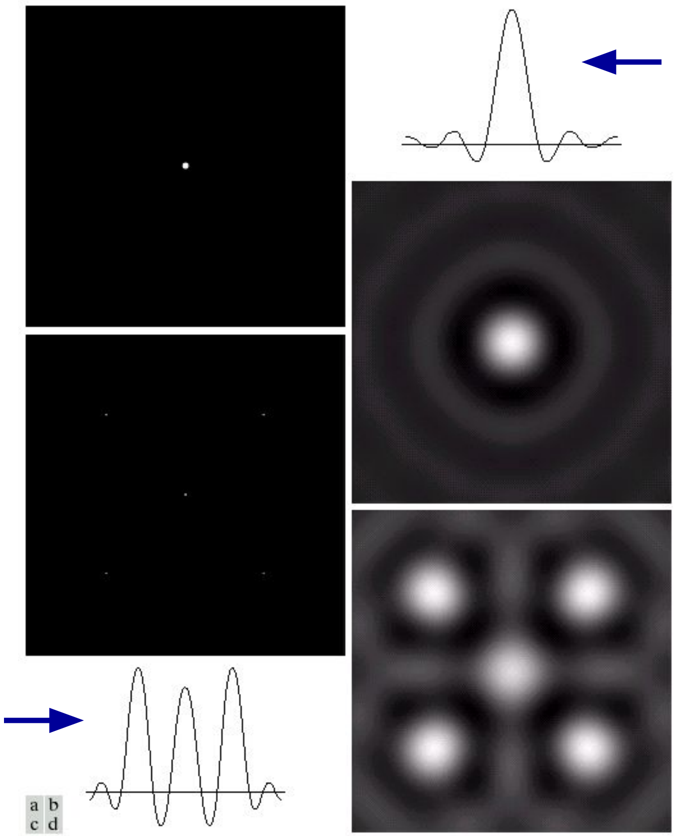
Less power removal

Less blurring

FIGURE 4.12 (a) Original image. (b)–(f) Results of ideal lowpass filtering with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). The power removed by these filters was 8, 5.4, 3.6, 2, and 0.5% of the total, respectively.

BLURRING AS A CONVOLUTION PROCESS

Obtained using the filter to its right.



Gray-scale profile of a **horizontal** scan line through the center of the spatial filter.

Gray-scale profile of a **diagonal** scan line through the center of the filtered image.

FIGURE 4.13 (a) A frequency-domain ILPF of radius 5. (b) Corresponding spatial filter (note the ringing). (c) Five impulses in the spatial domain, simulating the values of five pixels. (d) Convolution of (b) and (c) in the spatial domain.

BUTTERWORTH LOWPASS FILTERS

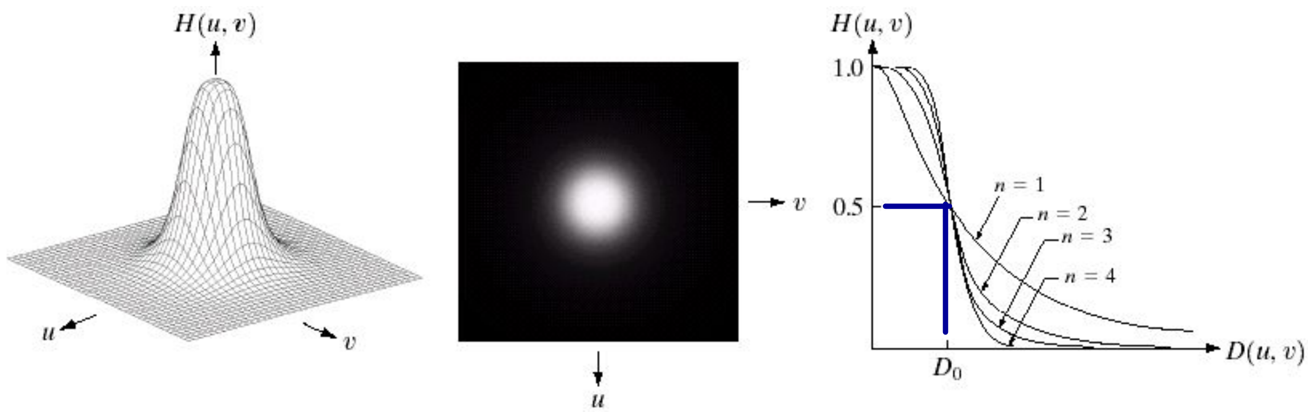
$$H(u,v) = \frac{1}{1 + [D(u,v)/D_0]^{2n}}$$

BLPF of order *n*

$$D(u,v) = [(u - M/2)^2 + (v - N/2)^2]^{1/2}$$

BLPF transfer function does not have a sharp discontinuity that establishes a clear cutoff.

BLPF WITH ORDERS 1 THROUGH 4

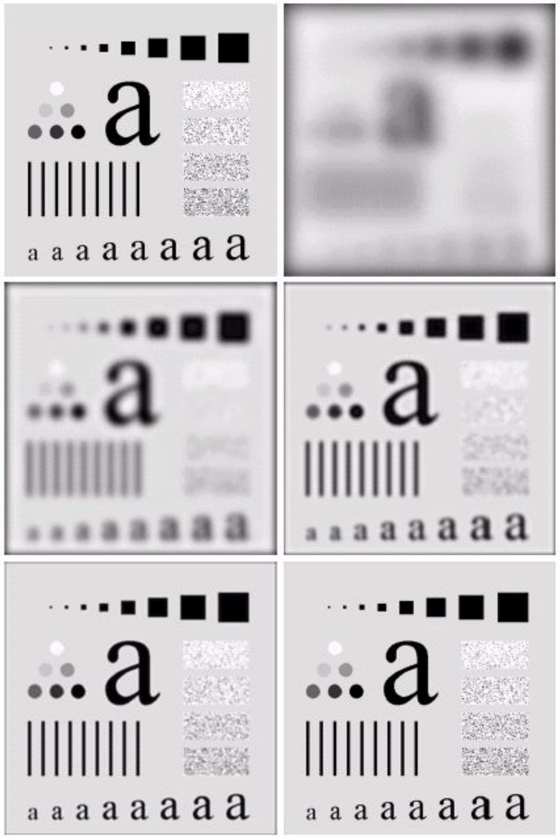


a b c

FIGURE 4.14 (a) Perspective plot of a Butterworth lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

BUTTERWORTH FILTERING WITH RADII 5,15,30,80,230

Ringing is not visible in any of these images.



a b
c d
e f

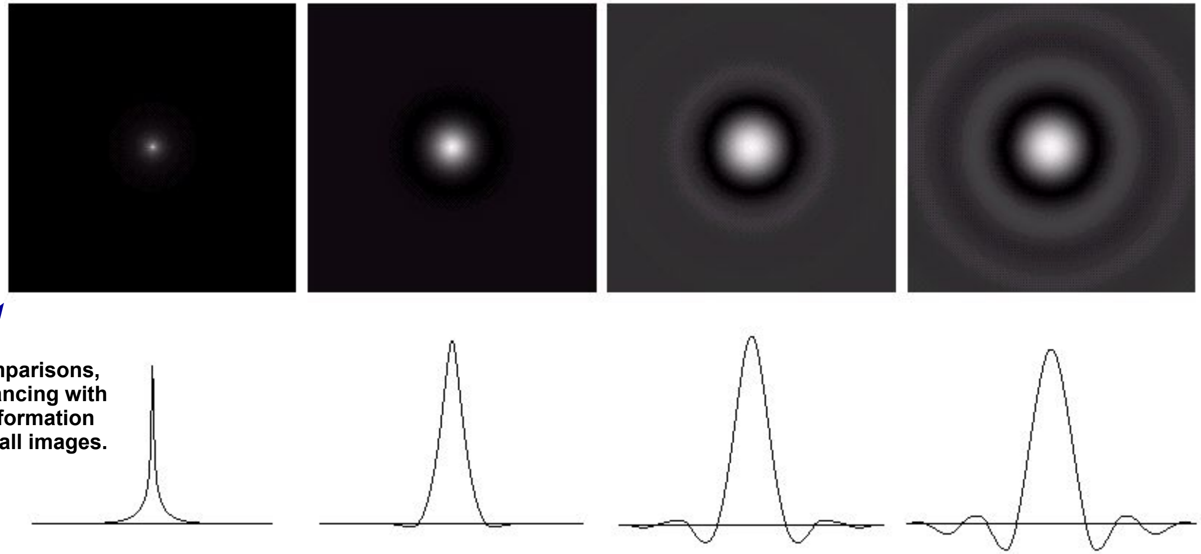
FIGURE 4.15 (a) Original image. (b)–(f) Results of filtering with BLPFs of order 2, with cutoff frequencies at radii of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Fig. 4.12.

BLPFS OF ORDER 1,2,5,20

no ringing
no negative values

mild ringing
small negative values

exhibits the characteristics
of the ILPF



To faciliate comparisons, additional enhancing with a gamma transformation was applied to all images.

a b c d

FIGURE 4.16 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding gray-level profiles through the center of the filters (all filters have a cutoff frequency of 5). Note that ringing increases as a function of filter order.

GAUSSIAN LOWPASS FILTERS

$$H(u,v) = e^{-D^2(u,v)/2\sigma^2}$$

σ is a measure of the spread of the Gaussian curve.

$$D(u,v) = [(u - M/2)^2 + (v - N/2)^2]^{1/2}$$

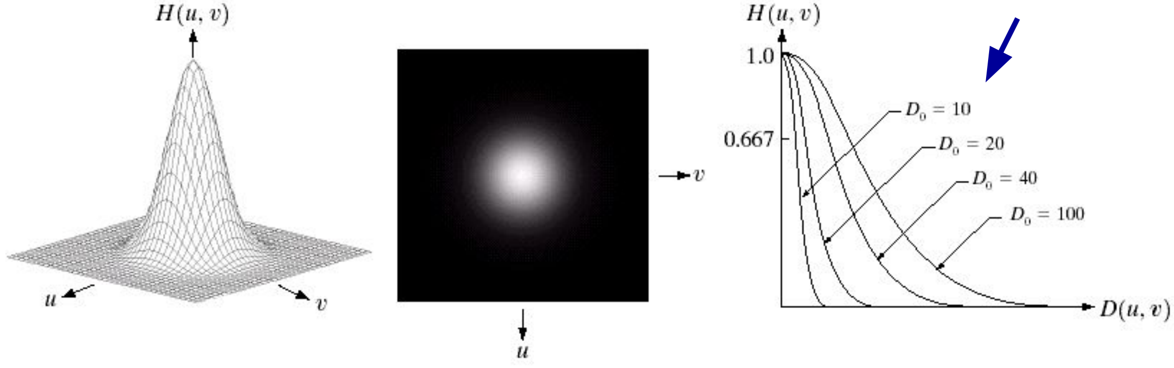
A **= 1 (to be consistent with the other filters)**

The inverse Fourier transform of the Gaussian lowpass filter is also Gaussian.

- **A spatial Gaussian filter will have no ringing.**

GLPFS WITH DIFFERENT σ VALUES

$\sigma = D_0$



a b c

FIGURE 4.17 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

GAUSSIAN FILTERING WITH RADII 5,15,30,80,230

no ringing

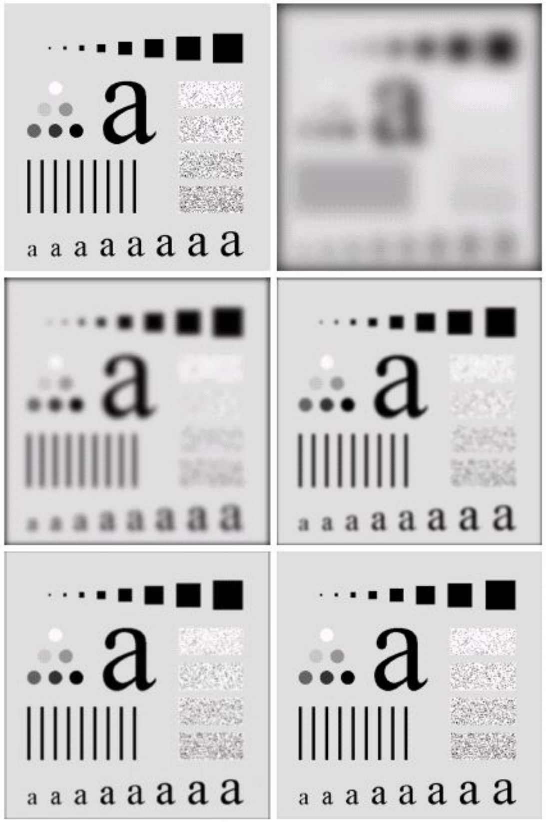
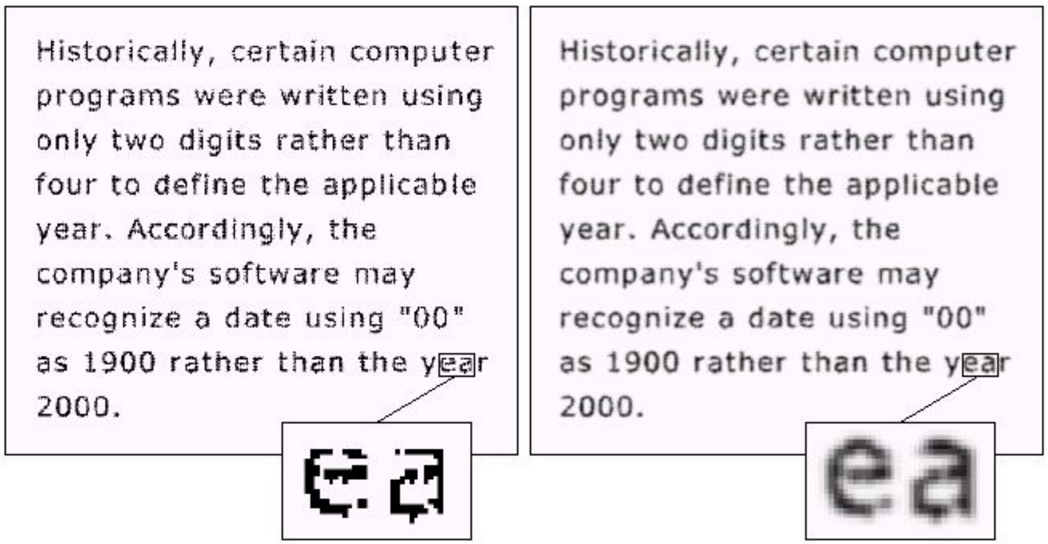


FIGURE 4.18 (a) Original image. (b)–(f) Results of filtering with Gaussian lowpass filters with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Figs. 4.12 and 4.15.

PRACTICAL APPLICATION OF LPF – MACHINE PERCEPTION

a b

FIGURE 4.19 (a) Sample text of poor resolution (note broken characters in magnified view). (b) Result of filtering with a GLPF (broken character segments were joined).



Machine recognition systems have difficulty in reading broken characters.

GLPF with D_0 = 80

PRACTICAL APPLICATION OF LPF – PRINTING & PUBLISHING

Smoothed image looks **soft** and **pleasing**!

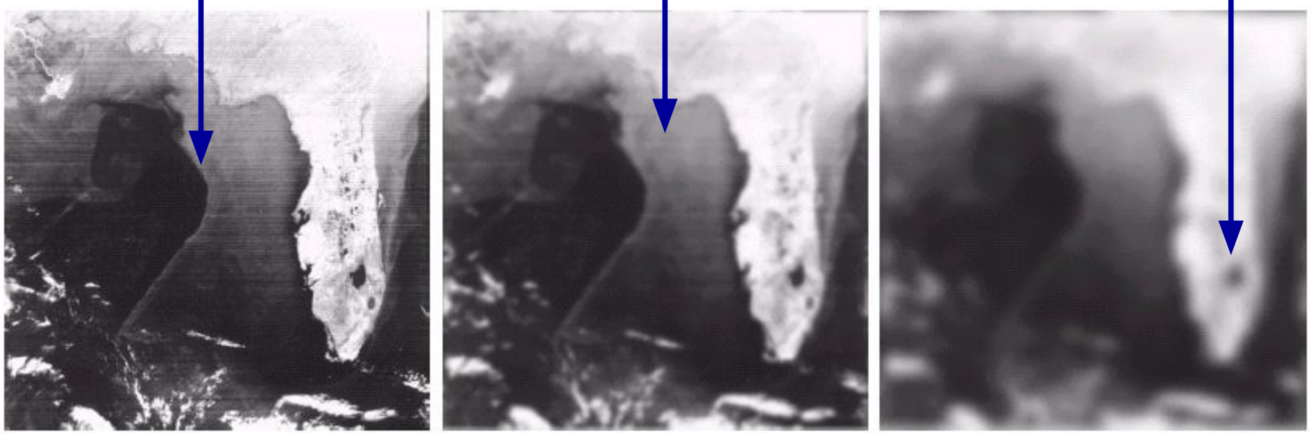
FIGURE 4.20 (a) Original image (1028 × 732 pixels). (b) Result of filtering with a GLPF with $D_0 = 100$. (c) Result of filtering with a GLPF with $D_0 = 80$. Note reduction in skin fine lines in the magnified sections of (b) and (c).

PRACTICAL APPLICATION OF LPF – PROCESSING SATELLITE AND AERIAL IMAGES

Boundaries between water bodies were caused by loop currents.

Scan lines are reduced.

More blurring to make large features recognizable..



a b c

FIGURE 4.21 (a) Image showing prominent scan lines. (b) Result of using a GLPF with $D_0 = 30$. (c) Result of using a GLPF with $D_0 = 10$. (Original image courtesy of NOAA.)

SHARPENING FREQUENCY DOMAIN FILTERS

Blurring is achieved by attenuating the **HF** components of DFT of an image.

Sharpening is achieved by attenuating the **LF** components of DFT of an image.

Only **zero-phase-shift** filters are considered here.

$$H_{hp}(u,v) = 1 - H_{lp}(u,v)$$

When LP attenuates frequencies, HP passes them.

Transfer function of the corresponding LP filter

Sharpening filters:

- **Ideal** highpass filters
- **Butterworth** highpass filters
- **Gaussian** highpass filters

3 TYPES OF SHARPENING FILTERS

The Butterworth filter represents a transition between the sharpness of the IF and the smoothness of the GF.

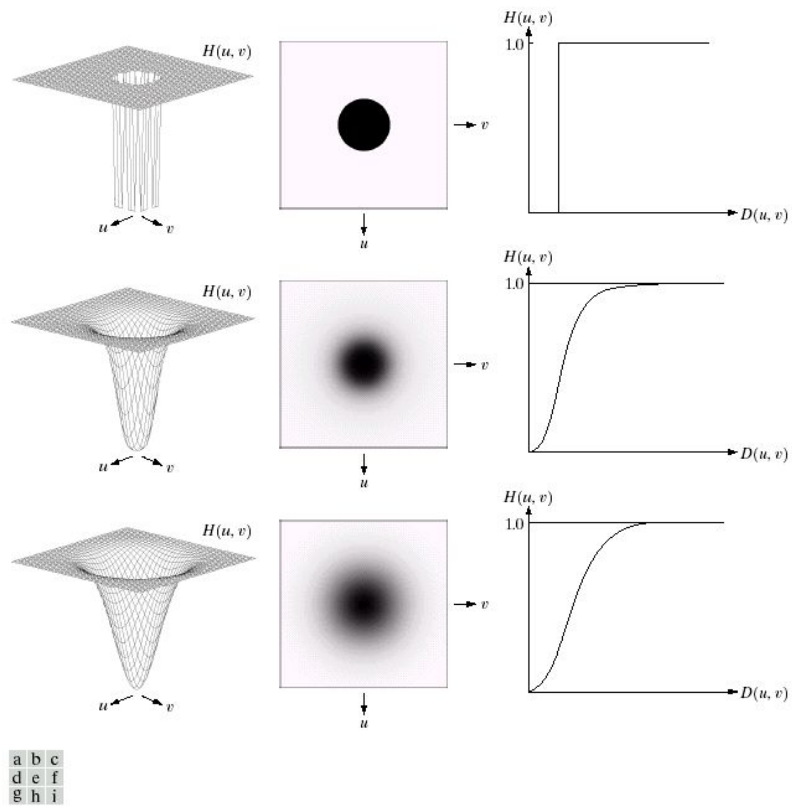
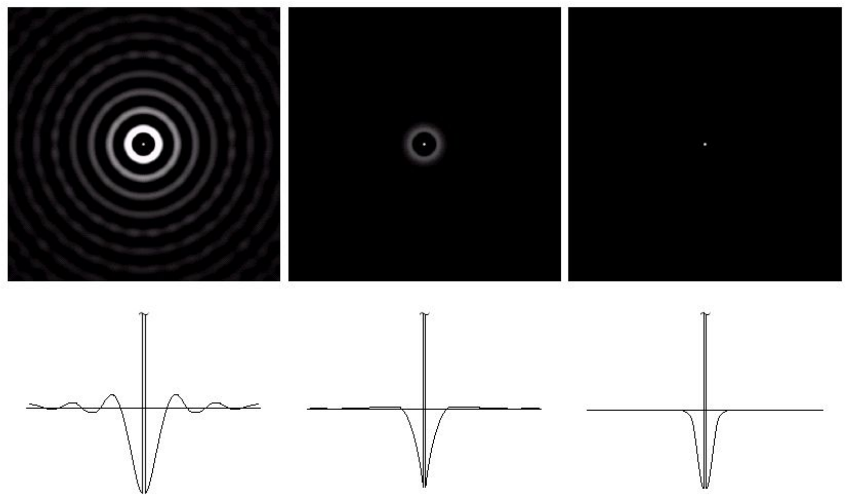


FIGURE 4.22 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

CORRESPONDING SPATIAL DOMAIN FILTERS



a b c

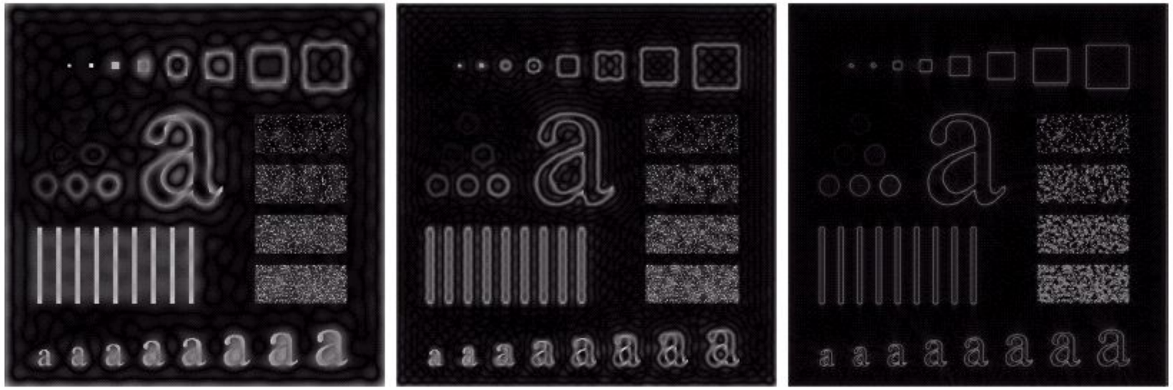
FIGURE 4.23 Spatial representations of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding gray-level profiles.

To obtain the spatial domain representation of a frequency domain filter:

1. **Multiply H(u,v) by (-1)u+v**
2. **Compute the inverse DFT**
3. **Multiply the real part by (-1)x+y**

IDEAL HIGHPASS FILTERS

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$

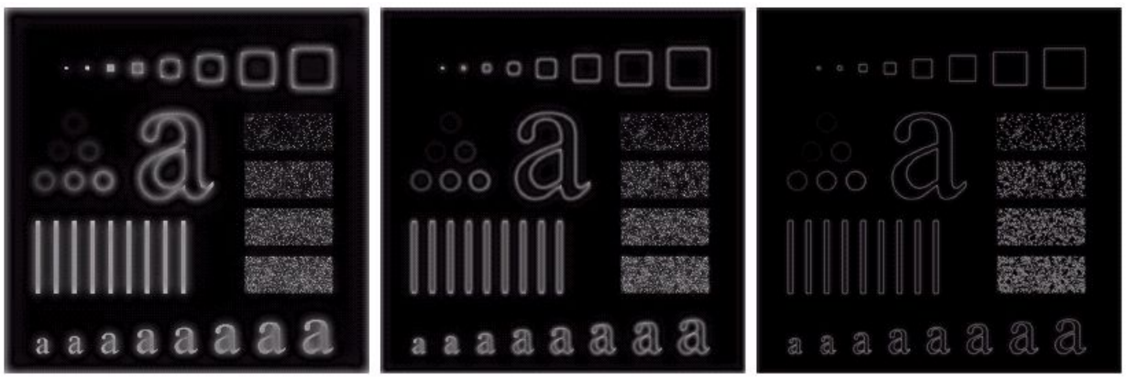


a b c

FIGURE 4.24 Results of ideal highpass filtering the image in Fig. 4.11(a) with $D_0 = 15$, 30, and 80, respectively. Problems with ringing are quite evident in (a) and (b).

BUTTERWORTH HIGHPASS FILTERS

$$H(u,v) = \frac{1}{1 + [D_0 / D(u,v)]^{2n}}$$



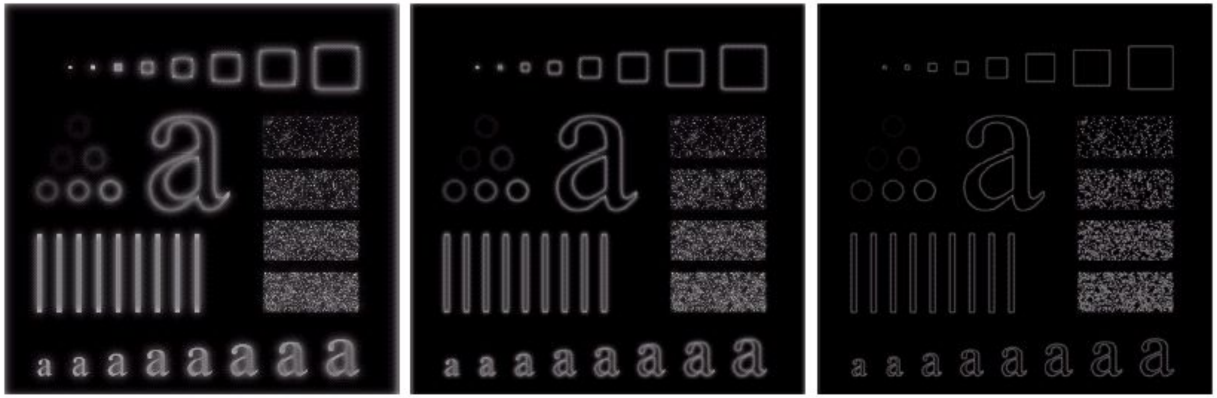
a b c

FIGURE 4.25 Results of highpass filtering the image in Fig. 4.11(a) using a BHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. These results are much smoother than those obtained with an ILPF.

GAUSSIAN HIGHPASS FILTERS

$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$$

results are smoother than with the previous 2 filters.



a b c

FIGURE 4.26 Results of highpass filtering the image of Fig. 4.11(a) using a GHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. Compare with Figs. 4.24 and 4.25.